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Hecke operators

1. Setup:

$\cdot)$ $\mathfrak{h} \ni \tau$: cplx. upper half plane.

$\cdot)$ $\mathcal{E} \xrightarrow{p} \mathfrak{h}$ ell. curve with fibers

$$\mathbb{C}\times\mathfrak{h}/\Lambda = \mathcal{E}_\tau = \mathbb{C}/\Lambda_\tau \,, \quad \Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\cdot\tau$$
$$\qquad\quad \Psi\, z$$

$\cdot)$ $\omega_{\mathfrak{h}} := p_* \Omega^1_{\mathcal{E}/\mathfrak{h}} \; (= \mathcal{O}_{\mathfrak{h}} \cdot dz).$

$\cdot)$ $GL_2^+(\mathbb{Q}) \curvearrowright \mathfrak{h}$ via Möb. trsf.

$\cdot)$ $SL_2(\mathbb{Z}) \curvearrowright \mathcal{E}$ via

$$\gamma\cdot(z,\tau) = (\, j(\gamma,\tau)^{-1}\cdot z \,, \; \gamma\cdot\tau \,)$$

Recall: $j(\gamma,\tau) = c\tau + d$, $\quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\Lambda = \{ (m + n\tau, \tau) \mid \tau \in \mathfrak{h}, \; m,n \in \mathbb{Z} \}.$$

$\cdot)\ j(\gamma, \tau)^{-1} \Lambda_\tau = \Lambda_{\gamma\cdot\tau}$

$\Lambda_\tau = \langle a\tau+b, c\tau+d \rangle$

$\cdot)\ j(\gamma\cdot\gamma', \tau) = j(\gamma, \gamma'\cdot\tau)\cdot j(\gamma', \tau)$ $\lrcorner$

$\Rightarrow \omega_h$ gib an $SL_2(\mathbb{Z})$ - equiv. structure.

$$\gamma^*(dz) = j(\gamma, \tau)^{-1} \cdot dz .$$

Extend this to $GL_2^+(\mathbb{Q})$ - equiv. str. by

$$\gamma^*(dz) = \underbrace{\det(\gamma)}_{\text{volume factor.}} \cdot j(\gamma, \tau)^{-1} \cdot dz .$$

·) Given $\Gamma \subseteq SL_2(\mathbb{Z})$, have

$$Y_\Gamma := \Gamma \backslash \mathfrak{h} \quad \text{(alg. curve)}.$$

X_Γ be its compactification.

equipped with $\mathcal{E}_\Gamma \xrightarrow{p} Y_\Gamma$,

$$\omega_\Gamma = p_* \Omega^1_{\mathcal{E}_\Gamma / Y_\Gamma}.$$

·) ω_Γ extends to X_Γ.

2. Kodaira-Spencer isom.

Prop: $\omega_{\mathfrak{h}}^{\otimes 2} \cong \Omega^1_{\mathfrak{h}/\mathbb{C}}$ as $SL_2(\mathbb{Z})$-equiv. l.b.

Pf: For $\gamma \in GL_2^+(\mathbb{Q})$:

$$\gamma^*(d\tau) = \frac{d}{d\tau}\left(\frac{a\tau + b}{c\tau + d}\right) \cdot d\tau$$

$$= \det(\gamma) \cdot j(\gamma, \tau)^{-2} \cdot d\tau.$$

$\Rightarrow \quad (2\pi i \, d_z)^{\otimes 2} \longmapsto 2\pi i \, d\tau$ is isom. $\square$

Cor: $\omega_\Gamma^{\otimes 2} \cong \Omega^1_{Y_\Gamma/\mathbb{C}}$ on Y_Γ.

Rmk:
1) Extends to $\omega_\Gamma^{\otimes 2} \cong \Omega^1_{X_\Gamma/\mathbb{C}}(\text{cusps})$ on X_Γ.

2) Isom. in Prop. is not $GL_2^+(\mathbb{Q})$-equiv.

3) k field, S/k smooth 1-dim, $p: E \to S$ ell. curve, $\omega := p_* \Omega^1_{E/S}$.

$\rightsquigarrow$ $KS: \omega^{\otimes 2} \longrightarrow \Omega^1_{S/k}$,
$a \otimes b \longmapsto \langle a, Db \rangle_{\text{Poinc}}$.

- not isom. in general.

We will identify (for $k \in \mathbb{Z}$)

$M_k(\Gamma) \longleftrightarrow H^0(Y_\Gamma, \mathcal{L}_\Gamma^{(k)})$, $f \mapsto f \cdot dz^{\otimes k-2} \cdot dz$.

$\mathcal{L}_\Gamma^{(k)} = \omega_\Gamma^{\otimes k-2} \otimes \Omega^1_{Y_\Gamma/\mathbb{C}}$.

3. Hecke operators:

<u>Def</u>: <u>Correspondence</u> (from X to Y) is a diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}, \quad f, g \text{ fin. étale}.$$

A <u>system of l.b.</u> is $(\mathcal{L}_X, \mathcal{L}_Y, \psi)$:

- $\mathcal{L}_X, \mathcal{L}_Y$ l.b. on X, Y
- $\psi: f^*\mathcal{L}_Y \xrightarrow{\sim} g^*\mathcal{L}_X$.

<u>Idea</u>: Correspondence = multi-valued function
"$X \to Y, \ x \mapsto f(g^{-1}(x))$".

Can pull back sections along corr:

$$H^0(Y, \mathcal{L}_Y) \xrightarrow{f^*} H^0(Z, f^*\mathcal{L}_Y) \xrightarrow{\sim} H^0(Z, g^*\mathcal{L}_X)$$

$$\xrightarrow[(*)]{Tr} H^0(X, \mathcal{L}_X)$$

(*): g fin. loc. free

$\rightsquigarrow \quad g_* \mathcal{O}_Z \xrightarrow{Tr} \mathcal{O}_X \quad \Big| - \otimes_{\mathcal{O}_X} \mathcal{L}_X$

$$g_* g^* \mathcal{L}_X \to \mathcal{L}_X .$$

If $s \in X(k)$ gen. pt., then we have

$$\begin{array}{ccc} (g_* g^* \mathcal{L}_X)(s) & \longrightarrow & \mathcal{L}_X(s) \\ \text{\rotatebox{90}{$\cong$}} & \nearrow & \\ \bigoplus_{s' \in g^{-1}(s)} \mathcal{L}_X(s) & & \sum_{s'} t_{s'} . \\ \rotatebox{90}{$\in$} & \longmapsto & \\ (t_{s'})_{s'} & & \end{array}$$

Def: $\Gamma_1, \Gamma_2 \subseteq SL_2(\mathbb{Z})$, $\alpha \in GL_2^+(\mathbb{Q})$.

Associated Hecke corr. :

$$\begin{tikzcd} & Y_{\Gamma_3} \arrow[dl] \arrow[dr, "\alpha"] & \\ Y_{\Gamma_2} & & Y_{\Gamma_1} \end{tikzcd} \qquad \Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2 \, ,$$

$$\left(\alpha : Y_{\Gamma_3} \xrightarrow{\sim} Y_{\alpha\Gamma_3\alpha^{-1}} \right).$$

Rmk: ·) NTC: Γ_3 cong. subgp.

·) Corr. only depends on $\Gamma_1 \alpha \Gamma_2$.

Now $\Omega^1_{Y_{\Gamma_i}/\mathbb{C}}$ and ω_{Γ_i} give syst. of l.b. on corr.

↳ use $GL_2^+(\mathbb{Q})$ equiv. str.

$\Rightarrow$ so does $\mathcal{L}^{(k)}_{\Gamma_i}$

$\Rightarrow$ Obtain pullback maps (Hecke operators):

$$H^0(Y_{\Gamma_1}, \mathcal{L}_{\Gamma_1}^{(k)}) \xrightarrow{[\Gamma_1 \alpha \Gamma_2]_k} H^0(Y_{\Gamma_2}, \mathcal{L}_{\Gamma_2}^{(k)}).$$

<u>Lemma:</u>

1) $$\Gamma_3 \backslash \Gamma_2 \cong \Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2,$$

$$\gamma \longmapsto \alpha\gamma.$$

2) Hecke operators are given by

$$f|_{[\Gamma_1 \alpha \Gamma_2]} = \sum_j f|_{[\beta_j]_k},$$

$$\Gamma_1 \alpha \Gamma_2 = \bigcup_j \Gamma_1 \beta_j,$$

$$(f|_{[\beta]_k})(z) = (\det \beta)^{k-1} j(\beta, z)^{-k} f(\beta(z))$$

for $\beta \in GL_2^+(\mathbb{Q})$.

Pf sketch of 2):

$$H^0(\mathcal{L}^{(k)}_{\Gamma_1}) \to H^0(\mathcal{L}^{(k)}_{\Gamma_3}) \xrightarrow{Tr} H^0(\mathcal{L}^{(k)}_{\Gamma_2}).$$

$$f \longmapsto f|[\alpha]_k \longmapsto \sum_j (f|[\alpha]_k)|[\gamma_j]_k$$

$$\Gamma_3 \backslash \Gamma_2 = \bigcup_j \Gamma_3 \gamma_j \,. \quad \square$$

Example: $\Gamma = \Gamma_1(N) = \{\gamma \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (N)\}$.

$Y_1(N) = Y_{\Gamma_1(N)} = \{(E,P) \mid E$ ell. curve, $P \in E$ order $N\}/\cong$

1) $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) = \{\gamma \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (N)\}$.

Then $[\Gamma \alpha \Gamma]_k = [\alpha]_k$ (depends only on $d \in (\mathbb{Z}/N\mathbb{Z})^*$).

We denote this Hecke operator by

$$\langle d \rangle : M_k(\Gamma) \xrightarrow{\sim} M_k(\Gamma).$$

"diamond operators"

Correspondence:

$$\begin{tikzcd} & Y_1(N) \arrow[dl, "id"'] \arrow[dr] & \\ Y_1(N) & & Y_1(N) \end{tikzcd} \qquad (E,P) \mapsto (E, d\cdot P).$$

2) $\alpha = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$ p prime.

Denote $[\Gamma \alpha \Gamma]_k$ by T_p.

$\Gamma_3 = \Gamma_1(N) \cap \{ \gamma \mid \gamma \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} (p) \}$.

$$Y_{\Gamma_3} = \{ (E,P,C) \mid P \text{ pt. of order } N,\ C \subseteq E \text{ cyclic of order } p,\ C \cap \langle P \rangle = 0 \}.$$

Corr:

$$\begin{tikzcd} & (E,P,C) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (E,P) & Y_{\Gamma_3} \arrow[dl] \arrow[dr] & (E/C, \overline{P}) \\ Y_1(N) & & Y_1(N) \end{tikzcd}$$

Lemma: Representatives for $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ are given by:

$$\beta_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \qquad 0 \leq j < p$$

$$\beta_\infty = \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \qquad (\text{only if } p \nmid N)$$

$$m, n \in \mathbb{Z}, \ mp - nN = 1.$$

Cor: $f \in M_k(\Gamma)$

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i \tau}$$

$$a_n(T_p f) = a_{np}(f) + 1_N(p) \cdot p^{k-1} a_{n/p}(\langle p \rangle f).$$

$$\left(1_N(p) = \begin{cases} 1 & p \nmid N \\ 0 & p \mid N \end{cases}, \quad a_{n/p} = 0 \ \text{ if } n/p \notin \mathbb{Z} \right).$$

Pf: $0 \le j < p$:

$$f|_{\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}}{}_k(\tau) = p^{k-1} \cdot p^{-k} \cdot f\left(\frac{\tau+j}{p}\right)$$

$$= \frac{1}{p} \cdot \sum_{n=0}^{\infty} a_n(f) \cdot e^{2\pi i n(\tau+j)/p}$$

$$= \frac{1}{p} \sum_{n=0}^{\infty} a_n(f)\, \zeta_p^{n \cdot j} \cdot q^{n/p}$$

Now use $\sum_{j=0}^{p-1} \zeta_p^{n \cdot j} = \begin{cases} p & p \mid n \\ 0 & p \nmid n \end{cases}$. $\square$

4. adelic setup:

Recall: $\cdot)$ $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ $\left(= \prod_p \mathbb{Z}_p \right)$

$\cdot)$ $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$

$\left(= \{ (a_p)_p \in \prod_p \mathbb{Q}_p \mid a_p \in \mathbb{Z}_p \ \forall' p \} \right).$

$\cdot)$ $G/\mathbb{Q}$ alg. gp., topologize $G(\mathbb{A}_f)$ by any closed emb. $G \hookrightarrow \mathbb{A}^n_{\mathbb{Q}}$.

Def: Let E/F ell. curve, F char 0, $\bar{F}/F$ alg. closure.

$T_f(E) := \varprojlim_n \underbrace{E[n](\bar{F})}_{\cong (\mathbb{Z}/n\mathbb{Z})^2}$:

free rk. 2 $\hat{\mathbb{Z}}$-mod. w/ cont. $Gal(\bar{F}/F)$-action.

$V_f(E) = T_f(E) \otimes_{\hat{\mathbb{Z}}} \mathbb{A}_f$

Def.:

- Isogeny of ell. curves $E \to E'$ is non-const. map of group var's.

- $Ell_{/F}$: cat. of ell. curves
- $Ell^0_{/F} := Ell_{/F} \otimes_{\mathbb{Z}} \mathbb{Q} \; \left(= Ell_{/F}[\text{isogeny}^{-1}] \right)$.

Idea:
- $T_\ell(E)$: info about torsion pts.
- $V_\ell(E)$ loses info, but is functorial on $Ell^0_{/F}$.

- Fact:

$$\left\{ (E, \Lambda) \;\middle|\; \begin{array}{l} E \in Ell^0_{/F} \\ \Lambda \subseteq V_\ell(E) \ \hat{\mathbb{Z}}\text{-lattice} \end{array} \right\}$$

$$\cong \quad Ell_{/F}.$$

Def: $K \subseteq GL_2(\mathbb{A}_f)$ cpct. open.

Define a moduli problem $Y_K / \mathbb{Q}$ by:

$$Y_K(F) = \left\{ (E, \bar{\xi}) \;\middle|\; E \in Ell^0_{/F} \,, \quad \bar{\xi} \in \underbrace{\mathrm{Isom}_{/\mathbb{A}_f}(\mathbb{A}_f^2, V_f(E)) / K}_{\text{Galois inv.}} \right\} \Big/ \cong .$$

Thm: K small enough.

$\Rightarrow Y_K$ representable by sm. (conn.) curve $/ \mathbb{Q}$.

$$Y_K(\mathbb{C}) \cong GL_2(\mathbb{Q}) \backslash \underbrace{\mathfrak{h}^{\pm}}_{\mathbb{C} \setminus \mathbb{R}} \times GL_2(\mathbb{A}_f) / K$$

$$(\mathbb{C}/\Lambda_\tau , \tilde{?}) \longmapsfrom (\tau, g)$$

$$Y_K(\mathbb{C}) \cong \coprod_i \Gamma_i \backslash \mathfrak{h} \,, \quad \Gamma_i \subseteq SL_2(\mathbb{Z})$$

cong. subgps.

$\underline{\text{Ex}}$: $K = K(N) = \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} (N) \}$

$\rightsquigarrow Y_K(S) = \{ (E, \eta_N) \mid E \in \mathcal{E}ll/_S ,$

$\alpha_N : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N] \}.$

$$Y_K(\mathbb{C}) = \bigsqcup_{\phi(N)} \Gamma(N) \backslash \mathfrak{h} .$$

·) For $g \in GL_2(\mathbb{A}_f)$ have map

$$g : Y_K \longrightarrow Y_{g^{-1}Kg} , (E, \bar{\eta}) \mapsto (E, \bar{\eta} g)$$

This gives right action $(Y_K)_K \circlearrowleft GL_2(\mathbb{A}_f)$.

·) $\mathcal{E}_K \xrightarrow{p} Y_K$ univ. ell. curve.

$$\omega_K = p_* \Omega^1_{\mathcal{E}_K / Y_K} .$$

$(\omega_K)_K$ give "system of l.b." on $(Y_K)_K$

+ action of $GL_2(\mathbb{A}_f)$.

In partic. for $K_1, K_2 \subseteq GL_2(\mathbb{A}_f)$ cpct. op.,

$g \in GL_2(\mathbb{A}_f)$.

$$\rightsquigarrow \quad \begin{tikzcd} & Y_{K_3} \arrow[dl] \arrow[dr, "g"] & \\ Y_{K_1} & & Y_{K_2} \end{tikzcd},$$

$$K_3 = K_1 \cap g K_2 g^{-1}.$$

$+$ Systems of l.b. ω_{K_i}, $\Omega^1_{Y_{K_i}/\mathbb{Q}}$.

$\Rightarrow$ Can define Hecke operators.

$$H^0(Y_{K_2}, \mathcal{L}^{(k)}_{K_2}) \longrightarrow H^0(Y_{K_1}, \mathcal{L}^{(k)}_{K_1}).$$

$$\cup\!|$$

$$M_k(K)$$