Hodge structures and Abelian schemes over $\mathbb{C}$

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These notes correspond to the talk given in the PhD Seminar on December 20 2021.
So far we have defined Shimura data and the corresponding Shimura varieties, and last week we had a quick intermezzo and discussed some theory around Abelian schemes. The main examples of Shimura varieties which we will want to study in the rest of the seminar are the so-called "Siegel modular varieties". In order to work with those, it turns out that we will need a really cool theorem of which the origins date back to Riemann. It relates a special case of Abelian schemes, namely (families of) complex Abelian varieties to certain (variations of) Hodge structures. Stating and proving this theorem, along with discussing some background about Hodge structures and complex Abelian varieties, is the goal of todays talk.

1 Hodge structures

We will first study Hodge structures, and relate them to the Deligne torus of Talk 4. We discuss some basic properties, and in particular, we define polarizations and variations of Hodge structures.

1.1 Definition and first properties

Let V be a real vector space and set $V(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} V$. One defines complex conjugation on this space by setting $\overline{z \otimes v} = \overline{z} \otimes v$.

Definition 1.1. A *Hodge decomposition* of V is a decomposition

$$V(\mathbb{C}) = \bigoplus_{p,q \in \mathbb{Z} \times \mathbb{Z}} V^{p,q}$$

such that $V^{q,p} = \overline{V^{p,q}}$. A *Hodge structure* is a real vector space together with a Hodge decomposition. We call $\{(p,q) : V^{p,q} \neq 0\}$ the *type* of the Hodge structure.

Hodge structures form a category, if we equip them with the following set of morphisms.

Definition 1.2. A *morphism of Hodge structures* is a linear map $V \to W$ sending $V^{p,q}$ to $W^{p,q}$ for all $p, q \in \mathbb{Z}$.

Now note that for all $n \in \mathbb{Z}$, we have that $\bigoplus_{p+q=n} V^{p,q} = \overline{\bigoplus_{p+q=n} V^{p,q}}$ which implies - using some Galois theory - that there is a subspace $V_n \subset V$ such that

$$V_n(\mathbb{C}) = \bigoplus_{p+q=n} V^{p,q}.$$

This defines a decomposition of V into disjoint subspaces.

Definition 1.3. We call $V = \bigoplus_n V_n$ the *weight decomposition* of V. If there is a choice of n such that $V = V_n$ then we say that V has *weight* n.

Definition 1.4. An *integral (respectively rational)* Hodge structure is a free $\mathbb{Z}$-module V of finite rank (respectively a $\mathbb{Q}$-vector space) together with a Hodge decomposition of $V(\mathbb{R})$ such that the weight decomposition is defined over $\mathbb{Q}$.

Definition 1.5. The *Hodge filtration* associated to a Hodge structure is the filtration

$$F^\bullet : \cdots \supset F^p \supset F^{p+1} \supset \cdots$$

where $F^p = \bigoplus_{r \geq p} V^{r,s} \subset V(\mathbb{C})$.

Example 1.1 ([4], Example 2.6)**.** Let $\mathbb{Q}(m)$ denote the Hodge structure of weight $-2m$ with underlying vector space $(2\pi i)^m \mathbb{Q}$, so

$$(\mathbb{Q}(m))(\mathbb{C}) = (2\pi i)^m \mathbb{Q} \otimes V^{-m,-m}$$

where $V^{-m,-m} = \mathbb{C}$. We define $\mathbb{Z}(m)$ and $\mathbb{R}(m)$ similarly.

1.2 A bit of representation theory

We can actually understand Hodge structures as certain representations. Recall from Talk 1 or [[3], sections 14e-f] that an algebraic group G over a field k is a *split torus* if it is isomorphic to a finite product of copies of $\mathbb{G}_m$, and it is a *torus* if it becomes a split torus after basechange to the separable closure k_{sep} of k. These are particular examples of algebraic groups which are multiplicative, and by [[3], Theorem 14.17] those are in correspondence with finitely generated $\mathbb{Z}$-modules together with an action of $\mathrm{Gal}(k_{sep}/k)$. By [[3], Theorem 14.22] the representations of such groups are in correspondence with the orbits of $\mathrm{Gal}(k_{sep}/k)$ acting on the character group. We will now specialize this to our case.

Construction 1.1 (Deligne torus)**.** Recall the Deligne torus $\mathbb{S}$ from Talk 4, which is obtained from $\mathbb{G}_m$ over $\mathbb{C}$ by restriction of scalars. We have that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ and $\mathbb{S}(\mathbb{C}) \cong \mathbb{G}_m \times \mathbb{G}_m \cong \mathbb{C}^* \times \mathbb{C}^*$. We choose the isomorphism such that $\mathbb{R} \to \mathbb{C}$ induces $\mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C}), z \mapsto (z, \bar{z})$. Now we have that complex conjugation is given by $\overline{(z_1, z_2)} = (\overline{z}_2, \overline{z}_1)$. Note that the characters of $\mathbb{S}_\mathbb{C}$ are the homomorphisms $(z_1, z_2) \mapsto z_1^r z_2^s$ with $r, s \in \mathbb{Z}$, so $X^*(\mathbb{S}) = \mathbb{Z} \times \mathbb{Z}$ with complex conjugation acting as $(p, q) \mapsto (q, p)$.

Now following [[4], page 17], note that given a representation $h : \mathbb{S} \to \mathrm{GL}(V)$ of $\mathbb{S}$ on a real vector space V, we have that $V \otimes \mathbb{C} = \bigoplus_{\chi \in X^*(T)} V_\chi$ where

$$V_\chi = \{v \in V \otimes \mathbb{C} : s \cdot v = \chi(s)v \quad \forall s \in \mathbb{S}\}$$

and such that h is fixed by the Galois action, i.e. $\sigma V_\chi = V_{\sigma\chi}$ for all characters χ, where σ is complex conjugation in this case. That is, giving a representation is the same as giving a $\mathbb{Z} \times \mathbb{Z}$-grading on $V(\mathbb{C})$ such that $\overline{V^{p,q}} = V^{q,p}$ for all p, q, i.e. to give a Hodge structure on V. Normalizing, h corresponds to the Hodge structure such that

$$h_{\mathbb{C}}(z_1, z_2)v = z_1^{-p} z_2^{-q} v$$

for $v \in V^{p,q}$.

Remark 1.1. The $\mathbb{R}$-linear map $C = h(i)$ is called the *Weil operator*. We note that C acts as i^{q-p} on $V^{p,q}$ and C^2 acts as $(-1)^n$ on V_n.

Example 1.2 ([4], Example 2.9)**.** The Hodge structure $\mathbb{Q}(m)$ corresponds to the homomorphism $h(z) = (z\bar{z})^m$.

1.3 Hodge structures of type $\{(-1, 0), (0, -1)\}$

The most important example of Hodge structures for us is the following, see also [[4], Example 2.4]. Let J be a complex structure on a real vector space V and define $V^{-1,0}$ and $V^{0,-1}$ as the $\pm i$ eigenspaces of J acting on $V(\mathbb{C})$, so that $V(\mathbb{C}) = V^{-1,0} \oplus V^{0,-1}$. This defines a Hodge structure of type $\{(-1, 0), (0, -1)\}$, which is of weight -1.

Remark 1.2 ([4], Example 2.8)**.** In fact, one can see that every Hodge structure of this type comes from a (unique) complex structure. Let $h : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(V)$ be a complex structure on V. Then the restriction $\mathbb{C}^* \to \mathrm{GL}(V)$ is a Hodge structure of type $(-1, 0), (0, -1)$ and the associated complex structure is the one defined by h.

Remark 1.3. So to give a rational Hodge structure of type $\{(-1, 0), (0, -1)\}$ amounts to giving a $\mathbb{Q}$-vector space V and a complex structure on $V(\mathbb{R})$ and to give an integral Hodge structure of type $\{(-1, 0), (0, -1)\}$ means to give a $\mathbb{C}$-vector space V together with a lattice in V.

Definition 1.6 ([4], Example 2.11 and 2.12)**.** Let (V, h) be a Hodge structure of type $(-1, 0), (0, -1)$. An alternating bilinear form $t : V \times V \to \mathbb{R}(1)$ is a *polarization* if $t_{\mathbb{R}}(Ju, Jv) = t_{\mathbb{R}}(u, v)$ for all $u, v \in V$ and $\frac{1}{2\pi i} t_{\mathbb{R}}(u, Ju) > 0$ for $u \neq 0$.

One can make a very similar definition replacing $\mathbb{R}(1)$ by $\mathbb{Z}(1)$ or $\mathbb{Q}(1)$. Now let S be a connected complex manifold and let V be a real vector space. Suppose that we have a Hodge structure h_s of type $\{(-1, 0), (0, -1)\}$ on V for each $s \in S$. Write $V_s^{p,q} = V_{h_s}^{p,q}$ and $F_s^p = F_{h_s}^p V$. Let $d = \frac{\dim(V)}{2}$ be the dimension of $V_s^{-1,0}$ (which is equal to the dimension of $V_s^{0,-1}$) and note that this is independent of the choice of $s \in S$.

Definition 1.7. The family of Hodge structures $(h_s)_{s\in S}$ on V is *continuous* if for fixed p and q, the subspace $V_s^{p,q}$ varies continuously with s, i.e. the map

$$S \to G_d(V(\mathbb{C})), s \mapsto V_s^{p,q}$$

is continuous (where G_d is the Grassmanian of d-dimensional subspaces of $V(\mathbb{C})$).

Definition 1.8. A continuous family of Hodge structures is *holomorphic* if the Hodge filtration varies holomorphically with s, that is:

$$S \to G_D(V(\mathbb{C})), s \mapsto F_s^\bullet$$

is holomorphic.

Here $D = \{d\}$ where $d = \dim F_s^1 V$ and $G_D(V(\mathbb{C}))$ is the set of flags

$$F : V(\mathbb{C}) \supset V^1 \supset 0$$

where V^1 is a subspace of $V(\mathbb{C})$ of dimension d.

Definition 1.9. A holomorphic family of Hodge structures of type $\{(-1,0),(0,-1)\}$ is called a *variation of Hodge structures.*

For S a nonconnected complex manifold, we define a variation of Hodge structures to be one on each connected component.

Remark 1.4. The definition above is made for families of Hodge structures of type $\{(-1,0),(0,-1)\}$. For families of Hodge structures of another type, one can define a variation of Hodge structures in a very similar way, but requiring an extra condition called "Griffiths transversality".

It turns out that these variations of Hodge structures are linked to Shimura data in the following way.

Proposition 1.1 ([4], Proposition 5.9)**.** *Let (G,X) be a Shimura datum. Then X has a unique structure of a complex manifold such that for every representation $\rho : G_{\mathbb{R}} \to GL(V)$ we have that $(V, \rho \circ h)_{h\in X}$ is a holomorphic family of Hodge structures. For this complex structure, each family $(V, \rho\circ h)_{h\in X}$ is a variation of Hodge structures and X is a finite disjoint union of Hermitian symmetric domains.*

Remark 1.5. Note that we have already seen that X has a structure of a complex manifold in some of the previous talks. This one coincides with the structure given by the above proposition.

2 Abelian varieties over $\mathbb{C}$ and complex tori

We now want to relate the Hodge structures of type $\{(-1,0),(0,-1)\}$ discussed above to complex Abelian varieties. These have a very nice description in terms of complex tori, which we discuss first, after which we state and prove Riemann's Theorem and give a version for families of Abelian varieties and variations of Hodge structures.

2.1 Complex tori

Recall the following definition.

Definition 2.1. A *lattice* in a real or complex vector space V is a $\mathbb{Z}$-module generated by an $\mathbb{R}$-basis for V. A *complex torus* is a complex manifold isomorphic to $\mathbb{C}^n/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}^n$.

Here, we make $\mathbb{C}^n/\Lambda$ into a complex manifold by endowing it with the quotient structure. Now recall the following definition.

Definition 2.2. Let M be a path connected and locally path connected topological space. Then a path connected and simply connected covering space (i.e. a locally trivial bundle with discrete fibers) is a *universal covering space* of M.

We have that $\mathbb{C}^n$ is the universal covering space of $M = \mathbb{C}^n/\Lambda$ and so $\pi_1(M, 0) = \Lambda$. This means that $H_1(M, \mathbb{Z}) \cong \Lambda$ and so

$$H^1(M, \mathbb{Z}) \cong \mathrm{Hom}(\Lambda, \mathbb{Z}).$$

There is actually a more general formula for this.

Proposition 2.1 ([4], Proposition 6.4)**.** *Let $M = \mathbb{C}^n/\Lambda$. There is a canonical isomorphism*

$$H^m(M, \mathbb{Z}) \to Hom(\wedge^m \Lambda, \mathbb{Z})$$

for all $m \in \mathbb{Z}_{\geq 1}$.

Proof. We start by noting that

$$\mathrm{Hom}(\wedge^m \Lambda, \mathbb{Z}) \cong \wedge^m \mathrm{Hom}(\Lambda, \mathbb{Z}).$$

We now use the result for $m = 1$ and claim that the pairing

$$\wedge^m \mathrm{Hom}(\Lambda, \mathbb{Z}) \cong \wedge^m H^1(M, \mathbb{Z}) \to H^m(M, \mathbb{Z})$$

induced by the cup product defines an isomorphism.
To prove this, note that the above pairing is certainly an isomorphism for the circle S^1 (as its cohomology groups are $\mathbb{Z}, \mathbb{Z}, 0, \cdots$). Now by the Künneth formula, the pairing is also an isomorphism for $(S^1)^{2n} \cong M$. This gives the result. □

Proposition 2.2 ([4], Proposition 6.5)**.** *Let $\Lambda \subset \mathbb{C}^n$ and $\Lambda' \in \mathbb{C}^m$ be two lattices. A linear map $\alpha : \mathbb{C}^n \to \mathbb{C}^m$ such that $\alpha(\Lambda) \subset \Lambda'$ defines a holomorphic map $\mathbb{C}^n/\Lambda \to \mathbb{C}^m/\Lambda'$ sending zero to zero. Moreover, every holomorphic map $\mathbb{C}^n/\Lambda \to \mathbb{C}^m/\Lambda'$ sending zero to zero is of this form for a unique α.*

Concerning the proof: the main work is in showing the last part (so that every holomorphic map sending zero to zero can be obtained from a unique α). This is done by using again that $\mathbb{C}^n$ is a universal covering space for $\mathbb{C}^n/\Lambda$, so that we can find a lift of any map, after which we use some complex analysis to show uniqueness.

Remark 2.1. Note that it also follows from this statement that every holomorphic map $\mathbb{C}^n/\Lambda \to \mathbb{C}^m/\Lambda'$ sending zero to zero is in fact a group homomorphism.

Now let $M = \mathbb{C}^n/\Lambda$ be a complex torus again. The isomorphism $\mathbb{R}\otimes\Lambda \cong \mathbb{C}^n$ gives us a complex structure J on $\mathbb{R}\otimes\Lambda$.

Definition 2.3. A *Riemann form* for M is an alternating form $\psi : \Lambda\times\Lambda \to \mathbb{Z}$ such that

$$\psi_{\mathbb{R}}(Ju, Jv) = \psi_{\mathbb{R}}(u, v)$$

for all $u, v \in V$ and

$$\psi_{\mathbb{R}}(u, Ju) > 0$$

for all $u \neq 0$.

Definition 2.4. We say that M is *polarizable* if there exists a Riemann form on it.

Note that the complex structure on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ endows $\Lambda \cong H_1(M, \mathbb{Z})$ with an integral Hodge structure of weight -1. A Riemann form for M is now nothing but a polarization of the integral Hodge structure on Λ.
There is the following theorem, proof of which can be found in [[5], Chapter 1] or [[6], Section 7.2.2].

Theorem 2.1 ([4], Theorem 6.7)**.** *The complex torus M is projective if and only if it is polarizable.*

Using Chow's theorem, we derive the following.

Corollary 2.1. *A polarizable complex torus is a projective algebraic variety and holomorphic maps of polarizable complex tori are regular.*

2.2 Riemann's theorem

Now recall the following definition.

Definition 2.5. An *Abelian variety A* over a field k is a connected projective algebraic variety over k together with a group structure given by regular maps.

Example 2.1. Elliptic curves over the complex numbers are Abelian varieties of dimension 1. Note that these "are" complex tori.

In fact, any complex Abelian variety "is" a torus. Let A be an Abelian variety over $\mathbb{C}$ and let T be the tangent space of A at zero. There is the exponential map $T \to A(\mathbb{C})$ defined by $\exp(v) = \phi_v(1)$ where $\phi_v : \mathbb{C} \to A$ is the unique holomorphic homomorphism such that $d\phi_v$ takes the unit tangent vector $(\partial/\partial t)_0$ to $v \in V$. (Its existence is somewhat subtle to prove.)

Proposition 2.3 ([5], page 1-2)**.** *The exponential map $T \to A(\mathbb{C})$ is a surjective homomorphism of Lie groups and the kernel is a lattice.*

Sketch of the proof. To see that exp is a homomorphism, let $x, y \in T$. Consider the map

$$\mathbb{C} \to A(\mathbb{C}), t \mapsto \exp(tx)\exp(ty)$$

then as A is an Abelian variety, this is a holomorphic homomorphism. The image of $(\partial/\partial t)_0$ under the associated tangent map is $x + y$. But on the other hand, we can also consider the map

$$\mathbb{C} \to A(\mathbb{C}), t \mapsto \exp(t(x + y))$$

and again, the image of $(\partial/\partial t)_0$ under the associated tangent map is $x + y$. Setting $t = 1$, by uniqueness, we need to have that $\exp(x)\exp(y) = \exp(x + y)$. Surjectivity follows from the fact that A is connected and $\exp(T)$ contains a neighborhood of the unit, which will be open and closed.
Now let Λ be the kernel of exp. We note that it is a discrete subgroup of T, as the exponential map is injective on small open neighborhoods of the unit, and the induced map $T/\Lambda \to A$ is holomorphic and an isomorphism of algebraic groups, and so the tangent map at the identity is also an isomorphism. Using the Inverse Function Theorem, the inverse is holomorphic at the unit, so its holomorphic everywhere, using the translations. And so A is isomorphic to T/Λ as complex manifolds, implying T/Λ is compact. But lattices are the only discrete subgroups of vector spaces with a compact quotient. And so Λ must be a lattice. $\square$

Corollary 2.2. *There is an equivalence of categories*

$$\{\textit{complex Abelian varieties}\} \to \{\textit{polarizable complex tori}\}.$$

Proof. Note that this functor is well defined, as we have seen that from an Abelian variety, we can construct a projective complex torus, which is polarizable by Theorem 2.1. Full faithfulness follows from Proposition 2.2 and Remark 2.1. By Corollary 2.1, any polarizable complex torus is an Abelian variety, which shows essential surjectivity. $\square$

Remark 2.2 (Following Section 1.1 of [2] and some insights by Manuel Hoff (thanks!))**.** The definition of a polarization of an Abelian variety given in last week's talk coincides with this one. Namely, for V a complex vector space, we set

$$\bar{V}_{\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{C}}(\bar{V}, \mathbb{C}) = \{\mathbb{C}\text{-semilinear maps } V \to \mathbb{C}\}.$$

We note that this is isomorphic to $V_{\mathbb{R}}^*$ by sending $f : V \to \mathbb{C}$ to the composition $V \xrightarrow{f} \mathbb{C} \xrightarrow{\mathrm{Im}} \mathbb{R}$.
Now for a lattice $\Lambda \subset V$ we define the dual lattice by

$$\Lambda^* = \{g \in V_{\mathbb{R}}^* : g(x) \in \mathbb{Z} \text{ for all } x \in \Lambda\}$$

and we define the dual torus by $(V/\Lambda)^* = \bar{V}_{\mathbb{C}}^*/\Lambda^*$.
Claim: This dual is the same as the dual $\mathrm{Pic}^0(V/\Lambda)$ of V/Λ as an Abelian

variety (which was defined last time).
Reason: We note that V is a universal cover of V/Λ and one can prove that in fact, line bundles on V/Λ are in correspondence to line bundles on V together with a suitable descent datum. One can now show that isomorphism classes of line bundles on V/Λ are in correspondence with pairs

$$\{(\lambda, \alpha) : \lambda \in \mathrm{NS}(V/\Lambda), \alpha : \Lambda \to S^1\}$$

where $\mathrm{NS}(V/\Lambda)$ is the Néron-Severi group, i.e. the cokernel of the inclusion $\mathrm{Pic}^0(V/\Lambda) \to \mathrm{Pic}(V/\Lambda)$. From this description, we see that

$$\mathrm{Pic}^0(V/\Lambda) = \{(\lambda, \alpha) \in \mathrm{Pic}(V/\Lambda) : \lambda = 0\} \cong \mathrm{Hom}(\Lambda, S^1) \cong V_{\mathbb{R}}^*/\Lambda^*$$

where the last inverse isomorphism is given by $g \mapsto (u \mapsto e^{2\pi i g(u)})$. So both duals agree.
Now: we have four different descriptions of $\mathrm{NS}(V/\Lambda)$:

1. Alternating 2-forms $\beta : \Lambda \times \Lambda \to \mathbb{Z}$ such that after restricting to V we have that $\beta(ix, iy) = \beta(x, y)$ for all $x, y \in V$.
2. Hermitian forms $\gamma : V \times V \to \mathbb{C}$ such that $\mathrm{Im}(\gamma)$ restricts to $\Lambda \times \Lambda \to \mathbb{Z}$.
3. Symmetric maps $V/\Lambda \to (V/\Lambda)^*$.
4. $\mathrm{Pic}(V/\Lambda)/\mathrm{Pic}^0(V/\Lambda)$

Here, the equivalence between the (2) and (1) is given by sending $\gamma : V \times V \to \mathbb{C}$ to $\mathrm{Im}(\gamma)|_{\Lambda\times\Lambda}$. The equivalence between the (4) and (3) is given by sending a line bundle L to ϕ_L (as introduced last week). Finally, the equivalence between (2) and (3) is given by sending $\gamma : V \times V \to \mathbb{C}$ to the map induced by $V \to \bar{V}_{\mathbb{C}}^*, x \mapsto \gamma(x, -)$.
One can show: Under the above equivalence, the ample line bundles in $\mathrm{Pic}(V/\Lambda)/\mathrm{Pic}^0(V/\Lambda)$ correspond to polarizations (in the sense of last week), which correspond to positive definite Hermitian forms in turn, and those precisely correspond to the alternating 2-forms that are Riemann forms.
Conclusion: The polarizations of the tori under consideration here are the same as the ones defined last week.

We are now finally ready to state and prove Riemann's theorem.

Theorem 2.2 (Riemann's Theorem, [4], Theorem 6.8)**.** *The functor from the category of Abelian varieties over $\mathbb{C}$ to the category of polarizable integral Hodge structures of type $(-1, 0), (0, -1)$ sending an Abelian variety A to $H_1(A, \mathbb{Z})$ is an equivalence of categories.*

Proof. We can factor this functor over the equivalence

$$\{\text{complex Abelian varieties}\} \to \{\text{polarizable complex tori}\}$$

of Corollary 2.2. The functor from polarizable complex tori to polarizable integral Hodge structures of type $(-1, 0), (0, -1)$ given by $M \mapsto H_1(M, \mathbb{Z})$ is fully faithful by Proposition 2.2 and essential surjectivity is not hard to see in this case. $\square$

One interesting consequence of this statement is the following. Let AV^0 be the category of complex Abelian varieties with morphisms given by the formula $\mathrm{Hom}_{AV^0}(A, B) = \mathrm{Hom}_{\text{Abelian varieties}}(A, B) \otimes \mathbb{Q}$.

Corollary 2.3 ([4], Corollary 6.9)**.** *The functor $A \mapsto H_1(A, \mathbb{Q})$ is an equivalence from AV^0 to the category of polarizable rational Hodge structures of type $(-1, 0), (0, -1)$.*

2.3 Families of Abelian varieties and variations of Hodge structures

There is a slight generalization of Theorem 2.2 for families of Abelian varieties and variations of Hodge structures, which we will study now. Let S be a connected topological manifold.

Definition 2.6. A *local system of $\mathbb{Z}$-modules on S* is a sheaf F on S that is locally isomorphic to the constant sheaf $\mathbb{Z}^n$ for $n \in \mathbb{Z}_{\geq 0}$.

Let F be a local system of $\mathbb{Z}$-modules on S and suppose that we have a Hodge structure h_s of type $\{(-1, 0), (0, -1)\}$ on $F_s \otimes \mathbb{R}$ for every $s \in S$.

Definition 2.7. We call F together with the Hodge structures a *variation of integral Hodge structures on S* if it defines a variation of Hodge structures on every open subset of S that trivializes F.

Definition 2.8. A *polarization* of a variation of Hodge structures $(F, (h_s))$ is a pairing $\psi : F \otimes F \to \mathbb{Z}$ such that ψ_s is a polarization of (F_s, h_s) for every $s \in S$.

Now let V be a nonsingular algebraic variety over $\mathbb{C}$. Recall the following definition from last week.

Definition 2.9. A *family of Abelian varieties over V* is a flat map $f : A \to V$ of nonsingular varieties which is regular, together with a regular multiplication map $A \times_V A \to A$ over V such that the fibers of f are Abelian varieties of constant dimension.

We can now state the following generalization of Theorem 2.2.

Theorem 2.3 ([4], Theorem 14.8)**.** *The functor from the category of families of Abelian varieties over V to the category of polarizable integral variations of Hodge structures of type $(-1, 0), (0, -1)$ on S defined by sending (A, f) to $(R^1 f_* \mathbb{Z})^\vee$ is an equivalence of categories.*

This is further discussed in [[1], 4.4.3].

References

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