Symmetric monoidal ∞-categories and operads

0. Overview:

- Recall (Un-)Straightening
- Define symmetric monoidal (s.m.) ∞-categories
- Define commutative algebra objects in a s.m. ∞-cat. $\Big\}$ $\rightsquigarrow$ operads
- Define module objects over such commutative algebra objects

Reference: Higher Algebra - Lurie

Notation: We write $\mathcal{C}$ for $N(\mathcal{C})$.

1. (Un-) Straightening:

Def: $p: \mathcal{E} \to \mathcal{C}$ map of ∞-categories.

- $f: x \to y$ in $\mathcal{E}$ is p-cocartesian if

$$\forall z \in \mathcal{E}: \quad \begin{tikzcd} \mathrm{Hom}_{\mathcal{E}}(y,z) \arrow[r, "f^*"'] \arrow[d, "p"] & \mathrm{Hom}_{\mathcal{E}}(x,z) \arrow[d, "p"] \\ \mathrm{Hom}_{\mathcal{C}}(p(y),p(z)) \arrow[r, "p(f)^*"] & \mathrm{Hom}_{\mathcal{C}}(p(x),p(z)) \end{tikzcd}$$

is a pullback square (in $\mathcal{S}$).

Informally:

$$\begin{tikzcd} \mathcal{E} \arrow[d, "p"'] \\ \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} x \arrow[rr] \arrow[dr, "f"] \arrow[dd] & & z \arrow[dd] \\ & y \arrow[ur, dashed, "\exists!"'] \arrow[dd] & \\ p(x) \arrow[rr] \arrow[dr] & & p(z) \\ & p(y) \arrow[ur] & \end{tikzcd}$$

•) $p\text{-CoCart} \subseteq \mathrm{Ar}(\mathcal{E}) = \mathrm{Fun}([1], \mathcal{E})$ full subcat. spanned by p-cocartesian maps.

Fact: $p\text{-CoCart} \xrightarrow{(\text{source}, p)} \mathcal{E} \underset{p, \mathcal{C}, \text{source}}{\times} \mathrm{Ar}(\mathcal{C})$ is fully faithful.

Informally: "p-cocartesian maps are uniquely determined by their source and their image in $\mathcal{C}$."

•) p is <u>cocartesian</u> if $p\text{-CoCart} \longrightarrow \mathcal{E} \times_{\mathcal{C}} \mathrm{Ar}(\mathcal{C})$ is essentially surjective.

•) $\mathrm{CoCart}(\mathcal{C}) \subseteq \mathrm{Cat}_{\infty /\mathcal{C}}$ (non-full) subcat. given as follows:

objects: $(p : \mathcal{E} \to \mathcal{C})$ s.th. p is cocartesian

maps: $(F : (\mathcal{E}, p) \to (\mathcal{E}', p'))$ s.th. F sends p-cocartesian maps in $\mathcal{E}$ to p'-cocartesian maps in $\mathcal{E}'$.

<u>Thm ((Un-)Straightening)</u>: Let $\mathcal{C} \in \mathrm{Cat}_\infty$.

$$\rightsquigarrow \quad \mathrm{CoCart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty).$$

<u>Rmk:</u>

•) Suppose $\mathcal{C} \in \mathrm{Cat}_1$. Then the above equivalence restricts to

$$\mathrm{CoCart}_1(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_1)$$

where $\mathrm{CoCart}_1(\mathcal{C}) \subseteq \mathrm{CoCart}(\mathcal{C})$ is the full subcategory spanned by those $(p : \mathcal{E} \to \mathcal{C})$ with $\mathcal{E} \in \mathrm{Cat}_1$:

- To $(p : \mathcal{E} \to \mathcal{C}) \in \mathrm{CoCart}_1(\mathcal{C})$ one associates

$$F : \mathcal{C} \longrightarrow \mathrm{Cat}_1$$

$$\bar{x} \longmapsto p^{-1}(\bar{x}) := [0] \times_{\bar{x}, \mathcal{C}, p} \mathcal{E}, \quad \text{pullback in } \mathrm{Cat}_1.$$

$$(\bar{f} : \bar{x} \to \bar{y}) \longmapsto \left(\begin{array}{l} p^{-1}(\bar{x}) \longrightarrow p^{-1}(\bar{y}) \\ x \mapsto (y, \text{ where } f : x \to y \text{ is} \\ \qquad \text{p-cocart. lift of } \bar{f}) \end{array} \right).$$

- To $F : \mathcal{C} \longrightarrow \mathrm{Cat}_1$ one associates $p : \mathcal{E} \to \mathcal{C}$ with $\mathcal{E}$ given by:

objects: $(\bar{x}, x)$, $\bar{x} \in \mathcal{C}$, $x \in F(\bar{x})$.

maps: $\mathrm{Hom}_{\mathcal{E}}((\bar{x}, x), (\bar{y}, y)) = \left\{ (\bar{f}, f) \;\middle|\; \begin{array}{l} \bar{f} : \bar{x} \to \bar{y} \text{ in } \mathcal{C} \\ f : F(\bar{f})(x) \to y \text{ in } F(\bar{y}) \end{array} \right\}$

•) If $p : \mathcal{E} \longrightarrow \mathcal{C}$ is an arbitrary cocart. map of ∞-cats., then the associated functor $F : \mathcal{C} \longrightarrow \mathrm{Cat}_\infty$ still satisfies $F(\bar{x}) = p^{-1}(\bar{x})$ for $\bar{x} \in \mathcal{C}$.

<u>Example:</u> Define the category Mod:

objects: (R, M), R ring, M R-module.

maps: $\mathrm{Hom}_{\mathrm{Mod}}((R, M), (S, N)) := \{ (\varphi, f) \mid \varphi : R \to S,\ f : M \to N \text{ } R\text{-linear} \}$.

Then $p : \mathrm{Mod} \to \mathrm{Rings}$, $(R, M) \mapsto R$ is cocartesian:

Given $(R, M) \in \mathrm{Mod}$, $\varphi : R \to S$, a p-cocartesian lift is given by

$$(R, M) \longrightarrow (S, S \otimes_R M), \quad (\varphi, m \mapsto 1 \otimes m)$$

Under Thm., p corresponds to the functor

$$\mathrm{Rings} \longrightarrow \mathrm{Cat}_1, \quad R \longmapsto \mathrm{Mod}(R), \quad (\varphi: R \to S) \longmapsto \left(S \otimes_R - : \mathrm{Mod}(R) \to \mathrm{Mod}(S) \right).$$

2. Symmetric monoidal ∞-categories:

Def:

- A commutative monoid is a pair $(M, \cdot)$ with M a set and $\cdot : M \times M \to M$ s.th.:
  - $\forall x, y \in M$: $x \cdot y = y \cdot x$
  - $\forall x, y, z \in M$: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
  - $\exists e \in M$: $\forall x \in M$: $e \cdot x = x$
- A symmetric monoidal 1-category is a tuple $(\mathcal{M}, \otimes, \beta, \gamma)$ consisting of
  - $\mathcal{M} \in \mathrm{Cat}_1$
  - $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$
  - natl' isos. $\beta = (\beta_{xy} : x \otimes y \cong y \otimes x)_{x,y}$, $\gamma = (\gamma_{xyz} : (x \otimes y) \otimes z \cong x \otimes (y \otimes z))_{x,y,z}$.

  s.th.
  - compatibilities of β, γ
  - $\exists (e, \alpha)$: $e \in \mathcal{M}$, $\alpha = (\alpha_x : e \otimes x \cong x)_x$ s.th. compatibilities.
- $\mathrm{SymMonCat}_1$ the category of s.m. 1-cats.

Example: $R \in \mathrm{Ring} \rightsquigarrow \mathrm{Mod}(R)$ is a s.m. 1-cat. wrt. $- \otimes_R -$.

Problem: If we want to define $\mathrm{SymMonCat}_\infty$, we need an efficient way of organizing "higher compatibilities".

Def:

•) Define the 1-category Fin_*:

objects: $\langle n \rangle = \{*, 1, \dots, n\}$ for $n \in \mathbb{Z}_{\geq 0}$.

maps: maps of pointed sets.

•) For $1 \leq i \leq n$, define $\rho_i^n : \langle n \rangle \to \langle 1 \rangle$, $\begin{array}{l} i \mapsto 1 \\ i \neq j \mapsto * \end{array}$.

Notation: Write $\langle n \rangle^\circ := \{1, \dots, n\} \subseteq \langle n \rangle$. We can think of maps $\langle m \rangle \to \langle n \rangle$ in Fin_* as partially defined maps of sets $\langle m \rangle^\circ \to \langle n \rangle^\circ$.

Construction: Let $(\mathcal{M}, \otimes, \beta, \gamma) \in SymMonCat_1$.

•) Define $X_{\mathcal{M}} : Fin_* \longrightarrow Cat_1$,
$$\left(\begin{array}{c} \langle n \rangle \longmapsto \mathcal{M}^n \\ (\alpha : \langle m \rangle \to \langle n \rangle) \mapsto \left(\begin{array}{c} \mathcal{M}^m \longrightarrow \mathcal{M}^n \\ (x_i)_{i=1}^m \mapsto \left(\bigotimes_{j \in \alpha^{-1}(i)} x_j \right)_{i=1}^n \end{array} \right) \end{array} \right).$$

•) Let $(\mathcal{M}^\otimes \to Fin_*) \in CoCart_1(Fin_*)$ be the unstraightening of $X_{\mathcal{M}}$.

Concretely this means that $\mathcal{M}^\otimes$ is given as follows:

objects: $(\langle n \rangle, x_1, \dots, x_n)$, $\langle n \rangle \in Fin_*$, $x_1, \dots, x_n \in \mathcal{M}$.

maps: $Hom_{\mathcal{M}^\otimes}((\langle m \rangle, x_1, \dots, x_m), (\langle n \rangle, y_1, \dots, y_n))$
$$= \left\{ (\alpha, f_1, \dots, f_n) \;\middle|\; \begin{array}{l} \alpha : \langle m \rangle \to \langle n \rangle \text{ in } Fin_* \\ f_i : \bigotimes_{j \in \alpha^{-1}(i)} x_j \to y_i \end{array} \right\}$$

This construction gives rise to a fully faithful functor

$$\mathrm{SymMonCat}_1 \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Cat}_1)$$

with essential image given by those functors $X : \mathrm{Fin}_* \longrightarrow \mathrm{Cat}_1$ s.th.

$$\forall n \in \mathbb{Z}_{\geq 0} : \quad (\rho^n_i)_{i=1}^n : X_n \xrightarrow{\sim} \prod_{i=1}^n X_1 \quad .$$

<u>Def:</u>

- A <u>s.m. ∞-cat.</u> $\mathcal{M}$ is a map of ∞-cats $X_{\mathcal{M}} : \mathrm{Fin}_* \to \mathrm{Cat}_\infty$ s.th.

$$\forall n \in \mathbb{Z}_{\geq 0} : \quad (\rho^n_i)_{i=1}^n : X_{\mathcal{M},n} \xrightarrow{\sim} (X_{\mathcal{M},1})^n \quad .$$

- $\mathrm{SymMonCat}_\infty \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Cat}_\infty)$ full subcat. spanned by s.m. ∞-cats.
- Given $X_{\mathcal{M}} \in \mathrm{SymMonCat}_\infty$ we have associated $(\mathcal{M}^\otimes \to \mathrm{Fin}_*) \in \mathrm{CoCart}(\mathrm{Fin}_*)$, $\mathcal{M} := X_{\mathcal{M},1} \in \mathrm{Cat}_\infty$. We call either of $X_{\mathcal{M}}, \mathcal{M}^\otimes, \mathcal{M}$ a s.m. ∞-cat.

<u>Example:</u>

- $\mathrm{SymMonCat}_1 \subseteq \mathrm{SymMonCat}_\infty$ is a full subcategory.
- Given an ∞-cat. with finite products, we can equip it with the cartesian symmetric monoidal structure:

- Define a 1-category $\Gamma^\times$ as follows:

objects: $(\langle n\rangle, T)$, $\langle n\rangle \in \mathrm{Fin}_*$, $T \subseteq \langle n\rangle^\circ$.

maps: $\mathrm{Hom}_{\Gamma^\times}((\langle m\rangle, S), (\langle n\rangle, T)) := \{\alpha : \langle m\rangle \to \langle n\rangle \mid \alpha^{-1}(T) \subseteq S\}$.

Then we have a natural functor $\Gamma^\times \to \mathrm{Fin}_*$ and the fibers are given by

$$\Gamma^\times_{\langle n\rangle} \cong \mathcal{P}(\langle n\rangle^\circ)^{op}$$ (where $\mathcal{P}(-)$ denotes the power set).

Now suppose we are given $\mathcal{C} \in \mathrm{Cat}_\infty$ with finite products.

- Define $\tilde{\mathcal{C}}^\times \in \mathrm{Set}_\Delta$ by setting

$$\tilde{\mathcal{C}}^\times_k := \coprod_{[k] \to \mathrm{Fin}_*} \mathrm{Hom}_{\mathrm{Set}_\Delta}([k] \times_{\mathrm{Fin}_*} \Gamma^\times, \mathcal{C}) \quad \text{for } k \in \mathbb{Z}_{\geq 0}.$$

Then we have a natural map $\tilde{\mathcal{C}}^\times \longrightarrow \mathrm{Fin}_*$ with fibers given by

$$\tilde{\mathcal{C}}^\times_{\langle n\rangle} \cong \underline{\mathrm{Hom}}_{\mathrm{Set}_\Delta}(\mathcal{P}(\langle n\rangle^\circ)^{op}, \mathcal{C}).$$

- Let $\mathcal{C}^\times \subseteq \tilde{\mathcal{C}}^\times$ be the full simplicial subset spanned by the 0-simplices $f : \mathcal{P}(\langle n\rangle^\circ)^{op} \longrightarrow \mathcal{C}$ s.th.

$$\forall S \subseteq \langle n\rangle^\circ : \quad f(S) \xrightarrow{\sim} \prod_{j \in S} f(\{j\}).$$

<u>Proposition:</u>

- $\mathcal{C}^\times$ is an ∞-category
- $(\mathcal{C}^\times \to \mathrm{Fin}_*)$ is a s.m. ∞-cat. with underlying ∞-cat. $\mathcal{C}$.

$\mathcal{C}^\times$ is called the <u>cartesian s.m. ∞-cat.</u> associated to $\mathcal{C}$.

3. Operads and Algebras:

<u>Def:</u> A <u>1-operad</u> $\mathcal{O}$ consists of the following data:

- A class of objects (write $a \in \mathcal{O}$)
- $\forall$ finite sets I, objects $(a_i)_{i \in I}, b$: a set of "multimaps"

$$\mathrm{Mul}_{\mathcal{O}}((a_i)_{i \in I}, b)$$

- $\forall$ map of finite sets $\alpha: I \to J$, objects $(a_i)_{i \in I}, (b_j)_{j \in J}, c$:

  a composition map

$$\mathrm{Mul}_{\mathcal{O}}((b_j)_{j \in J}, c) \times \prod_{j \in J} \mathrm{Mul}_{\mathcal{O}}((a_i)_{i \in \alpha^{-1}(j)}, b_j) \to \mathrm{Mul}_{\mathcal{O}}((a_i)_{i \in I}, c).$$

We want these data to satisfy the following conditions:

- Associativity
- Identity maps.

We obtain a category Op_1 of 1-operads. Also, given $\mathcal{O}, \mathcal{O}' \in \mathrm{Op}_1$ we have a functor category $\mathrm{Fun}(\mathcal{O}, \mathcal{O}') \in \mathrm{Cat}_1$.

<u>Remark:</u> We have a forgetful functor $\mathrm{Op}_1 \to \mathrm{Cat}_1$ and an inclusion $\mathrm{Cat}_1 \subseteq \mathrm{Op}_1$ that make up an adjunction

$$\mathrm{Cat}_1 \rightleftarrows \mathrm{Op}_1.$$

Example:

- Let $(\mathcal{M}, \otimes, \beta, \gamma) \in \mathrm{SymMonCat}_1$. Then we can define a 1-operad $\tilde{\mathcal{M}}$ having the same objects as $\mathcal{M}$ and with maps given by

$$\mathrm{Mul}_{\tilde{\mathcal{M}}}\left((x_i)_{i\in I}, y\right) := \mathrm{Hom}_{\mathcal{M}}\left(\bigotimes_{i \in I} x_i, y\right).$$

This gives a (non-full!) inclusion $\mathrm{SymMonCat}_1 \subseteq \mathrm{Op}_1$.

- Comm: objects: a

maps: $\mathrm{Mul}_{\mathrm{Comm}}\left((a)_{i\in I}, a\right) = \{*\}$

- $\mathrm{Comm}^{[1]}$: objects: a, b

maps: $$\mathrm{Mul}_{\mathrm{Comm}^{[1]}}\left((x_i)_{i\in I}, y\right) = \begin{cases} \{*\} & \forall i: x_i = a,\ y = a \\ \{*\} & y = b \\ \emptyset & \text{else} \end{cases}$$

- CommMod: objects: a, m

maps: $$\mathrm{Mul}_{\mathrm{CommMod}}\left((x_i)_{i\in I}, y\right) = \begin{cases} \{*\} & \forall i: x_i = a,\ y = a \\ \{*\} & \exists!\, i: x_i = m,\ y = m \\ \emptyset & \text{else} \end{cases}$$

We also have various maps between the last three of these examples:

$$\begin{tikzcd}
\mathrm{Comm} \arrow[rr, shift left, "b"] \arrow[rr, shift right, "a"'] \arrow[dr, "a"'] & & \mathrm{Comm}^{[1]} \\
& \mathrm{CommMod} \arrow[ur, "{a \mapsto a,\ m \mapsto b}"'] &
\end{tikzcd}$$

Def: Let $\mathcal{O} \in \mathrm{Op}_1$, $\mathcal{M} \in \mathrm{SymMonCat}_1$. An <u>$\mathcal{O}$-algebra in $\mathcal{M}$</u> is a map of 1-operads $\mathcal{O} \to \tilde{\mathcal{M}}$. We set $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) := \mathrm{Fun}(\mathcal{O}, \tilde{\mathcal{M}})$.

Remark: Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map in Op_1 and let $\mathcal{M} \in \mathrm{SymMonCat}_1$. Then we have a restriction functor $\mathrm{Alg}_{\mathcal{O}_2}(\mathcal{M}) \to \mathrm{Alg}_{\mathcal{O}_1}(\mathcal{M})$.

Example: Let $\mathcal{M} := (\mathrm{Ab}, \otimes)$. Then we have:

- $\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{M}) \cong \mathrm{Rings}$
- $\mathrm{Alg}_{\mathrm{Comm}^{[1]}}(\mathcal{M}) \cong \{\varphi : A \to B \mid A, B \in \mathrm{Rings}\} = \mathrm{Ar}(\mathrm{Rings})$
- $\mathrm{Alg}_{\mathrm{CommMod}}(\mathcal{M}) \cong \{(R, M) \mid R \in \mathrm{Rings},\ M\ R\text{-module}\} = \mathrm{Mod}$.

Now we also want to define $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$ when $\mathcal{M} \in \mathrm{SymMonCat}_\infty$.

Construction: Let $\mathcal{O} \in \mathrm{Op}_1$.

- Define a 1-category $\mathcal{O}^{\otimes}$:

  objects: $(\langle n \rangle, a_1, \dots, a_n)$, $\langle n \rangle \in \mathrm{Fin}_*$, $a_1, \dots, a_n \in \mathcal{O}$

  maps: $\mathrm{Hom}_{\mathcal{O}^{\otimes}}((\langle m \rangle, a_1, \dots, a_m), (\langle n \rangle, b_1, \dots, b_n))$

  $$= \left\{ (\alpha, f_1, \dots, f_n) \;\middle|\; \begin{array}{l} \alpha : \langle m \rangle \to \langle n \rangle \text{ in } \mathrm{Fin}_* \\ f_i \in \mathrm{Mul}_{\mathcal{O}}((a_j)_{j \in \alpha^{-1}(i)}, b_i) \end{array} \right\}.$$

- We have a natural map $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$.

  (note: For $\mathcal{M} \in \mathrm{SymMonCat}_1$, $\tilde{\mathcal{M}}^{\otimes} \cong \mathcal{M}^{\otimes}$, compatible with map to Fin_*).

[**Def**: $\alpha : \langle m \rangle \to \langle n \rangle$ in Fin_* is inert if $|\alpha^{-1}(i)| = 1 \ \forall i \in \langle n \rangle^\circ$]

•) $\mathcal{O}^\otimes \to Fin_*$ is in general not cocartesian, but inert maps always have cocartesian lifts:

Given $(\langle m \rangle, a_1, \dots, a_m) \in \mathcal{O}^\otimes$ and $\alpha : \langle m \rangle \to \langle n \rangle$ inert, such a lift is given by $(\alpha, id, \dots, id) : (\langle m \rangle, a_1, \dots, a_m) \to (\langle n \rangle, a_{\alpha^{-1}(1)}, \dots, a_{\alpha^{-1}(n)})$.

•) Given $\mathcal{O}_1, \mathcal{O}_2 \in Op_1$ we have a fully faithful inclusion

$$Fun(\mathcal{O}_1, \mathcal{O}_2) \subseteq Fun_{Fin_*}(\mathcal{O}_1^\otimes, \mathcal{O}_2^\otimes)$$

with essential image given by those functors preserving cocartesian maps lying over inert maps in Fin_*.

(compare this to the statement that $SymMonCat_1 \subseteq Op_1$ is not full).

<u>Remark</u>: One can give a definition of ∞-operad as a map of ∞-cats. $\mathcal{O}^\otimes \to Fin_*$ that is modelled on the above construction.

<u>Def</u>: Let $\mathcal{O} \in Op_1$, $\mathcal{M} \in SymMonCat_\infty$. Then we define $Alg_{\mathcal{O}}(\mathcal{M}) \subseteq Fun_{Fin_*}(\mathcal{O}^\otimes, \mathcal{M}^\otimes)$ as the full subcategory spanned by those maps $(\mathcal{O}^\otimes, p_{\mathcal{O}}) \to (\mathcal{M}^\otimes, p_{\mathcal{M}})$ that send $p_{\mathcal{O}}$-cocartesian maps in $\mathcal{O}^\otimes$ that lie over inert maps in Fin_* to $p_{\mathcal{M}}$-cocartesian maps in $\mathcal{M}$. We call objects in $Alg_{\mathcal{O}}(\mathcal{M})$ <u>$\mathcal{O}$-algebras in $\mathcal{M}$</u>.

Remark:

- This definition extends the definition of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$ for $\mathcal{M} \in \mathrm{SymMonCat}_1$ and in particular does not lead to a clash of notation.
- As before we have contravariant functoriality of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$ with respect to maps of 1-operads.

We are now in a place where we can effortlessly define the following notions (for $\mathcal{M} \in \mathrm{SymMonCat}_\infty$):

- $\mathrm{CAlg}(\mathcal{M}) := \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{M})$: commutative algebra objects in $\mathcal{M}$
- Given $A \in \mathrm{CAlg}(\mathcal{M})$:
  - $\mathrm{CAlg}(A) := [0] \underset{A, \mathrm{CAlg}(\mathcal{M}), a}{\times} \mathrm{Alg}_{\mathrm{Comm}^{[1]}}(\mathcal{M})$: commutative A-algebras.
  - $\mathrm{Mod}(A) := [0] \underset{A, \mathrm{CAlg}(\mathcal{M})}{\times} \mathrm{Alg}_{\mathrm{CommMod}}(\mathcal{M})$: A-modules.

The map of 1-operads $\mathrm{CommMod} \longrightarrow \mathrm{Comm}^{[1]}$ induces a natural forgetful functor

$$\mathrm{CAlg}(A) \longrightarrow \mathrm{Mod}(A).$$